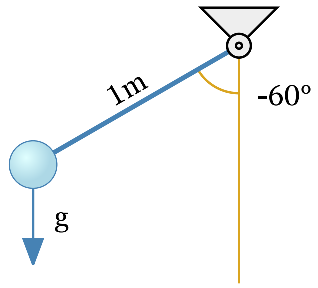
Comparison of different methods for analysis of simple undamped pendulum with Calcpad

Input parameters



Gravitational acceleration (m/s²) - $g = 9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}$

Pendulum length - $l = 1\,\mathrm{m}$

Pendulum mass - $m = 1\,\mathrm{kg}$

Initial angle - $\theta_0 = -60° = -1.05\ \mathrm{rad}$

Maximum simulation time - $t_{max} = 10\,\mathrm{s}$

Analytical solution for small rotations

$\theta \ll 1$ or $\sin(\theta) \approx \theta$

Differential equation - $\theta'' + \frac{g}{l} \cdot \theta = 0$

Angular frequency - $\omega = \sqrt{\frac{g}{l}} \cdot \mathrm{rad} = 3.13\ \mathrm{rad/s}$

Cyclic frequency - $f = \frac{\omega}{2 \cdot \pi \cdot \mathrm{rad}} = 0.498\ \mathrm{Hz}$

Period - $T = \frac{1}{f} = 2.01\ \mathrm{s}$

Equation of motion - $\theta(t) = \theta_0 \cdot \cos(\omega \cdot t)$

Analytical solution for large rotations (exact)

Differential equation - $\theta'' + \frac{g}{l} \cdot \sin(\theta) = 0$

Incomplete elliptic integral of the first kind

$$F(\varphi; k) = \int_0^{\varphi} \frac{1}{\sqrt{1 - k^2 \cdot \sin(\theta)^2}}\, \mathrm{d}\theta$$

Jacobi elliptic functions

Modulus - $k = \sin\left(\frac{\theta_0}{2}\right) = -0.5$

$am(u; k) = \$\mathrm{Root}\{F(\varphi; k) = u;\ \varphi \in [0; 10 \cdot \pi]\}$

$sn(u; k) = \sin(am(u; k))$, $cn(u; k) = \cos(am(u; k))$

$dn(u; k) = \sqrt{1 - k \cdot sn(u; k)^2}$, $cd(u; k) = \frac{cn(u; k)}{dn(u; k)}$

Period - $T_e = 4 \cdot \sqrt{\frac{l}{g}} \cdot F\left(\frac{\pi}{2}; k\right) = 2.15\ \mathrm{s}$

Error - $\delta_T = \frac{|T - T_e|}{T_e} = 6.82\ \%$

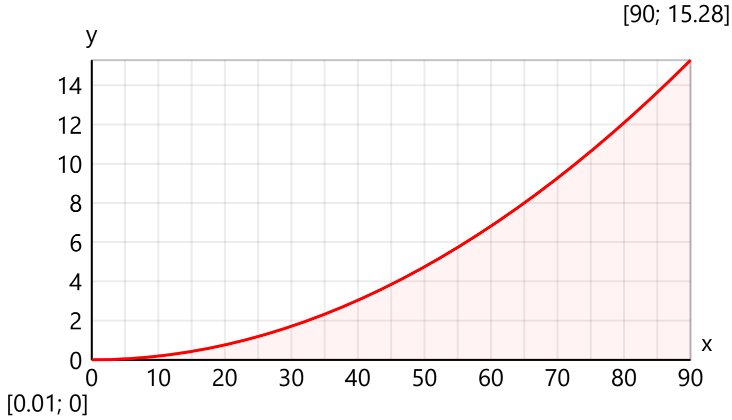
Cyclic frequency - $f_e = \frac{1}{T_e} = 0.464\ \mathrm{Hz}$

Angular frequency - $\omega_e = 2 \cdot \pi \cdot \mathrm{rad} \cdot f_e = 2.92\ \mathrm{rad/s}$

Equation of motion - $\theta_e(t) = 2 \cdot \mathbf{asin}\left(k \cdot cd\left(\sqrt{\frac{g}{l}} \cdot t; k\right)\right)$

Energy - $E_0 = m \cdot l \cdot g \cdot (1 - \mathbf{cos}(\theta_0)) = 4.9\ \mathrm{J}$

Relative error δ_T [%] of small displacements period versus initial angle θ_0 [°] plot



Solution by forward Euler method (explicit)

For that purpose, the II order equation is reduced the following system of I order equations

$\theta' = \omega$ and $\omega' = \frac{g}{l} \cdot \mathbf{sin}(-\theta)$

The solution is performed iteratively

Step size - $h = 0.05\ \mathrm{s}$

Number of steps - $n = \frac{t_{max}}{h} = 200$

For each time step $n = 200$ the values for the next step will be obtained by using the following equstions

$$\left|\begin{array}{l} \theta_{n+1} = \theta_n + h \cdot \omega_n \\ \omega_{n+1} = \omega_n + h \cdot \frac{g}{l} \cdot \mathbf{sin}\,\theta_n \end{array}\right.$$

Allocate vectors

$\vec{\theta}_{fwE} = \mathbf{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

$\vec{\omega}_{fwE} = \mathbf{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

$\vec{E}_{fwE} = \mathbf{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

Set initial conditions

$\vec{\theta}_{fwE.1} = \frac{\theta_0}{1\,\mathrm{rad}} = -1.05$, $\vec{\omega}_{fwE.1} = \frac{0}{\mathrm{s}}$

Perform Euler steps

Rotation - $\vec{\theta}_{fwE.2} = \vec{\theta}_{fwE.1} + h \cdot \vec{\omega}_{fwE.1} = -1.05$

Angular velicity - $\vec{\omega}_{fwE.2} = \vec{\omega}_{fwE.1} + h \cdot \frac{g}{l} \cdot \sin(-\vec{\theta}_{fwE.1}) = 0.425\ \mathrm{s}^{-1}$

Energy - $\vec{E}_{fwE.1} = m \cdot l^2 \cdot \left(\frac{1}{2} \cdot \vec{\omega}_{fwE.1}{}^2 + \frac{g}{l} \cdot (1 - \cos(\vec{\theta}_{fwE.1})) \right) = 4.9\ \mathrm{J}$

Solution by backward Euler method (implicit)

The following iterative procedure is applied:

$$\left| \begin{array}{l} \theta_{n+1} = \theta_n + h \cdot \omega_{n+1} \\ \omega_{n+1} = \omega_n + h \cdot \frac{g}{l} \cdot \sin\theta_{n+1} \end{array} \right.$$

Allocate vectors

$\vec{\theta}_{bwE} = \mathbf{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

$\vec{\omega}_{bwE} = \mathbf{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

$\vec{E}_{bwE} = \mathbf{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

Set initial conditions

$\vec{\theta}_{bwE.1} = \frac{\theta_0}{1\,\mathrm{rad}} = -1.05$, $\vec{\omega}_{bwE.1} = \frac{0}{\mathrm{s}}$

Perform Euler steps

$f(\theta) = \vec{\theta}_{bwE.i} + h \cdot \left(\vec{\omega}_{bwE.i} + h \cdot \frac{g}{l} \cdot \sin(-\theta) \right) - \theta$

Rotation - $\vec{\theta}_{bwE.2} = \$\mathrm{Root}\{f(\theta) = 0;\ \theta \in [-2 \cdot \pi;\ 2 \cdot \pi]\} = -1.03$

Angular velicity - $\vec{\omega}_{bwE.2} = \vec{\omega}_{bwE.1} + h \cdot \frac{g}{l} \cdot \sin(-\vec{\theta}_{bwE.2}) = 0.419\ \mathrm{s}^{-1}$

Energy - $\vec{E}_{bwE.1} = m \cdot l^2 \cdot \left(\frac{1}{2} \cdot \vec{\omega}_{bwE.1}{}^2 + \frac{g}{l} \cdot (1 - \cos(\vec{\theta}_{bwE.1})) \right) = 4.9\ \mathrm{J}$

Solution by Crank–Nicolson method (IMEX)

The following iterative procedure is applied:

$$\left| \begin{array}{l} \theta_{n+1} = \theta_n + \frac{h}{2} (\omega_n + \omega_{n+1}) \\ \omega_{n+1} = \omega_n + \frac{h \cdot g}{2 \cdot l} (\sin\theta_n + \sin\theta_{n+1}) \end{array} \right.$$

Allocate vectors

$\vec{\theta}_{CN} = \mathbf{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

$\vec{\omega}_{CN} = \mathbf{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

$\vec{E}_{CN} = \mathbf{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

Set initial conditions

$\vec{\theta}_{CN.1} = \frac{\theta_0}{1\,\mathrm{rad}} = -1.05$, $\vec{\omega}_{CN.1} = \frac{0}{\mathrm{s}}$

Perform Euler steps

$f(\theta) = \vec{\theta}_{CN.i} + \frac{h}{2} \cdot \left(2 \cdot \vec{\omega}_{CN.i} + \frac{h \cdot g}{2 \cdot l} \cdot (\sin(-\vec{\theta}_{CN.i}) + \sin(-\theta)) \right) - \theta$

Rotation - $\vec{\theta}_{CN.2} = \$Root\{f(\theta) = 0;\ \theta \in [-2 \cdot \pi;\ 2 \cdot \pi]\} = -1.04$

Angular velicity - $\vec{\omega}_{CN.2} = \vec{\omega}_{CN.1} + \frac{h \cdot g}{2 \cdot l} \cdot (\sin(-\vec{\theta}_{CN.1}) + \sin(-\vec{\theta}_{CN.2})) = 0.423\ s^{-1}$

Energy - $\vec{E}_{CN.1} = m \cdot l^2 \cdot \left(\frac{1}{2} \cdot \vec{\omega}_{CN.1}{}^2 + \frac{g}{l} \cdot (1 - \cos(\vec{\theta}_{CN.1}))\right) = 4.9\ J$

Solution by Runge-Kutta RK4 method (explicit)

The following iterative procedure is applied:

First step (k1) - $k_{1,\theta} = \omega_i$, $k_{1,\omega} = \frac{g}{l} \cdot \sin(-\theta_i)$

Second step (k2) - $k_{2,\theta} = \omega_i + 0.5 \cdot h \cdot k_{1,\omega}$, $k_{2,\omega} = \frac{g}{l} \cdot \sin(-(\theta_i + 0.5 \cdot h \cdot k_{1,\theta}))$

Third step (k3) - $k_{3,\theta} = \omega_i + 0.5 \cdot h \cdot k_{2,\omega}$, $k_{3,\omega} = \frac{g}{l} \cdot \sin(-(\theta_i + 0.5 \cdot h \cdot k_{2,\theta}))$

Fourth step (k4) - $k_{4,\theta} = \omega_i + h \cdot k_{3,\omega}$, $k_{4,\omega} = \frac{g}{l} \cdot \sin(-(\theta_i + h \cdot k_{3,\theta}))$

Update values using weighted averages

$$\theta_{n+1} = \theta_i + \frac{h}{6} \cdot (k_{1,\theta} + 2 \cdot k_{2,\theta} + 2 \cdot k_{3,\theta} + k_{4,\theta})$$
$$\omega_{n+1} = \omega_i + \frac{h}{6} \cdot (k_{1,\omega} + 2 \cdot k_{2,\omega} + 2 \cdot k_{3,\omega} + k_{4,\omega})$$

Allocate vectors

$\vec{\theta}_{RK4} = \mathrm{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

$\vec{\omega}_{RK4} = \mathrm{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

$\vec{E}_{RK4} = \mathrm{vector}(n) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots\ 0]$

Set initial conditions

$\vec{\theta}_{RK4.1} = \frac{\theta_0}{1\,rad} = -1.05$, $\vec{\omega}_{RK4.1} = \frac{0}{s}$

Perform Runge-Kutta 4 steps

RK4 factors

$k_{1,\theta} = \vec{\omega}_{RK4.1} = 0\ s^{-1}$, $k_{1,\omega} = \frac{g}{l} \cdot \sin(-\vec{\theta}_{RK4.1}) = 8.49\ s^{-2}$

$k_{2,\theta} = \vec{\omega}_{RK4.1} + 0.5 \cdot h \cdot k_{1,\omega} = 0.212\ s^{-1}$

$k_{2,\omega} = \frac{g}{l} \cdot \sin(-(\vec{\theta}_{RK4.1} + 0.5 \cdot h \cdot k_{1,\theta})) = 8.49\ s^{-2}$

$k_{3,\theta} = \vec{\omega}_{RK4.1} + 0.5 \cdot h \cdot k_{2,\omega} = 0.212\ s^{-1}$

$k_{3,\omega} = \frac{g}{l} \cdot \sin(-(\vec{\theta}_{RK4.1} + 0.5 \cdot h \cdot k_{2,\theta})) = 8.47\ s^{-2}$

$k_{4,\theta} = \vec{\omega}_{RK4.1} + h \cdot k_{3,\omega} = 0.423\ s^{-1}$

$k_{4,\omega} = \frac{g}{l} \cdot \sin(-(\vec{\theta}_{RK4.1} + h \cdot k_{3,\theta})) = 8.44\ s^{-2}$

Update values using weighted averages

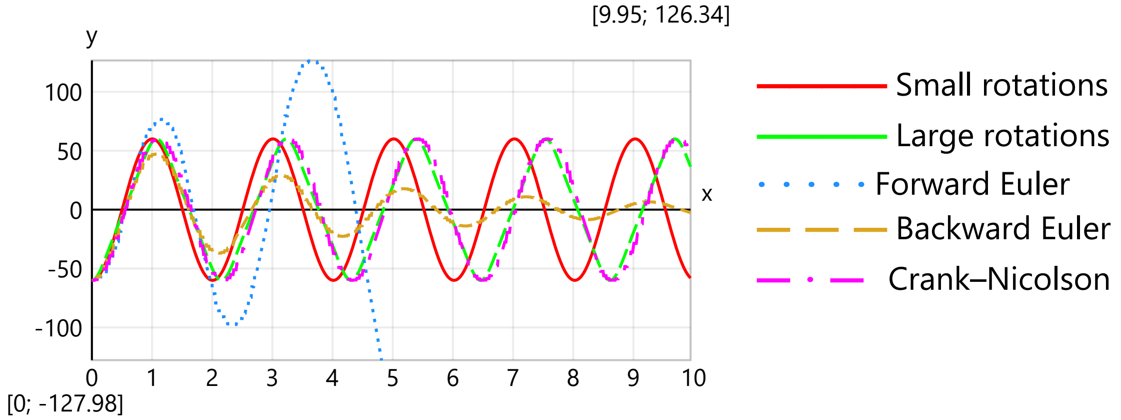
Rotation - $\vec{\theta}_{RK4.2} = \vec{\theta}_{RK4.1} + \frac{h}{6} \cdot (k_{1,\theta} + 2 \cdot k_{2,\theta} + 2 \cdot k_{3,\theta} + k_{4,\theta}) = -1.04$

Angular velicity - $\vec{\omega}_{RK4.2} = \vec{\omega}_{RK4.1} + \frac{h}{6} \cdot (k_{1,\omega} + 2 \cdot k_{2,\omega} + 2 \cdot k_{3,\omega} + k_{4,\omega}) = 0.424\ s^{-1}$

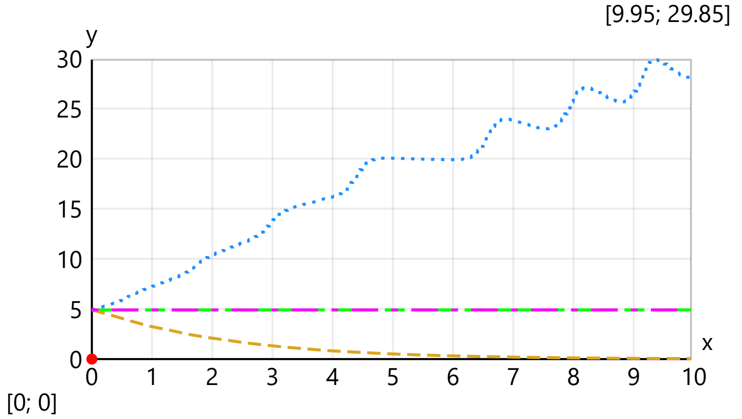
Energy - $\vec{E}_{RK4.1} = m \cdot l^2 \cdot \left(\frac{1}{2} \cdot \vec{\omega}_{RK4.1}{}^2 + \frac{g}{l} \cdot (1 - \cos(\vec{\theta}_{RK4.1}))\right) = 4.9\ J$

Plot results

Rotation θ [deg] versus time t [s] plot

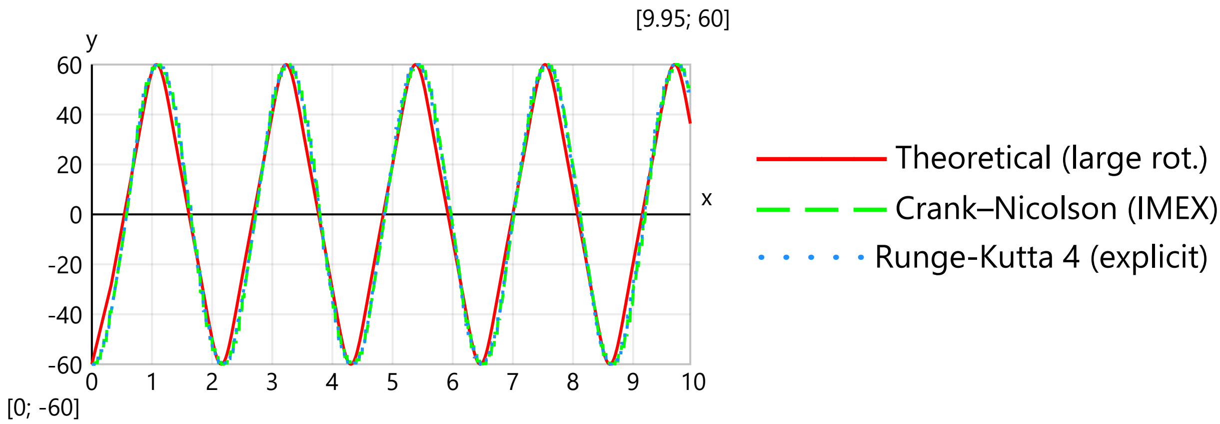


Energy E [J] versus time t [s] plot

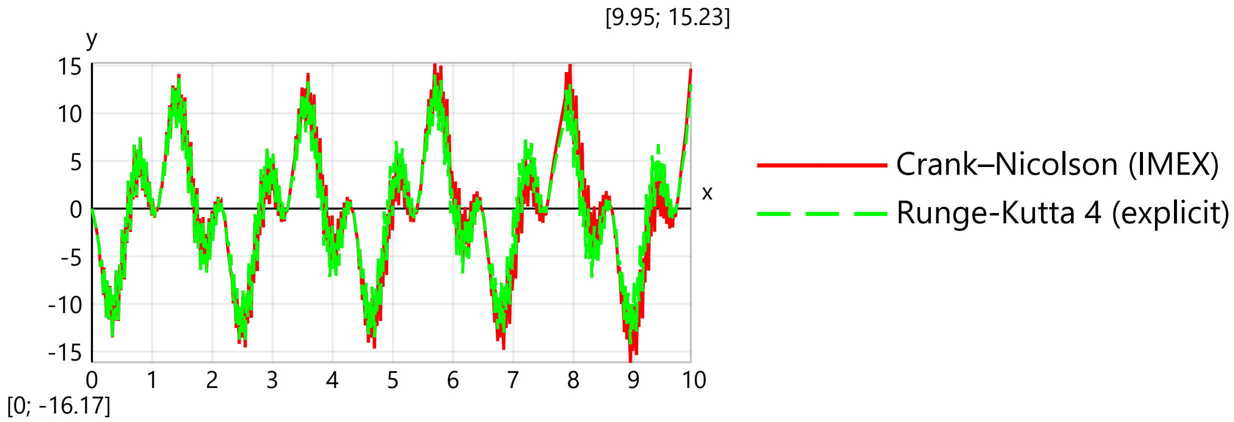


Comparision of Crank–Nicolson and Runge-Kutta 4 methods

Rotation θ [deg] versus time t [s] plot



Absolute error Δθ [°] versus time t [s] plot



Energy E [J] versus time t [s] plot